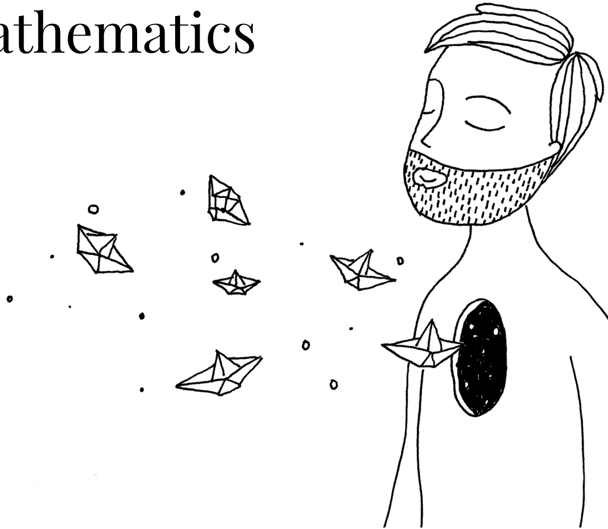
4509 - Bridging Mathematics

Lecture 1: Metric Spaces, Sequences and Topology



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Objectives

- Define a metric space (X, d) and see why generalized distances matter in economics (Euclidean distance in $\mathbb{R}^n$, the sup norm on $C(X)$, the ℓ_p spaces).
- Characterize open, closed, interior, closure and boundary sets.
- Master sequences in $\mathbb{R}^n$, Cauchy sequences, and completeness.

Metric spaces

Definition

A **metric space** is a pair (X, d) with $X \neq \emptyset$ and $d : X \times X \to \mathbb{R}_+$ such that for all x, y, z:

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ (symmetry);
3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The triangle inequality is the one tool we use all week: a detour is never shorter than the direct route.

Open and closed balls

Fix a metric d. The **open** and **closed** balls of radius $r > 0$ around x_0 are

$$B(x_0, r) = \{x : d(x, x_0) < r\}, \qquad \bar{B}(x_0, r) = \{x : d(x, x_0) \leq r\}.$$

A ball collects every point within distance r of x_0. Its shape depends on the metric, which we see next.

Metrics you must know

On $\mathbb{R}^n$:

- Euclidean (ℓ_2): $d_2(x, y) = \sqrt{\sum_i (x_i - y_i)^2} = \|x - y\|_2$.
- Manhattan (ℓ_1): $d_1(x, y) = \sum_i |x_i - y_i|$.
- Chebyshev (ℓ_∞): $d_\infty(x, y) = \max_i |x_i - y_i|$.

On function spaces:

- Sup norm on $C[a, b]$: $d_\infty(f, g) = \sup_x |f(x) - g(x)|$. This is where value functions will live.

Norm: the special case

Definition

A **norm** on a vector space V is a function $\|\cdot\| : V \to \mathbb{R}_+$ such that for all $x, y \in V$ and scalars α:

1. $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$;
2. $\|\alpha x\| = |\alpha| \, \|x\|$ (homogeneity);
3. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality).

Every norm gives a metric through $d(x, y) = \|x - y\|$: the norm is the distance to the origin, $\|x\| = d(x, 0)$, and the distance between two points is the norm of their difference.

Not every metric comes from a norm (the discrete metric does not): a norm needs a vector space, a metric needs only a set.

Inner product (Rynne and Youngson, Definition 3.1)

Definition

On a real vector space X, an **inner product** $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ satisfies, for all x, y, z and scalars α, β:

1. $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$;
2. $\langle \alpha x + \beta y,\ z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ (linear in the first argument);
3. $\langle x, y \rangle = \langle y, x \rangle$ (symmetric).

It induces the norm $\|x\| = \sqrt{\langle x, x \rangle}$. On $\mathbb{R}^n$ the standard inner product is $\langle x, y \rangle = \sum_i x_i y_i$, which returns the Euclidean norm, $\|x\|^2 = \langle x, x \rangle$.

Cauchy-Schwarz, and the triangle inequality

Theorem (Cauchy-Schwarz)

For x, y in an inner product space, $\langle x, y\rangle^2 \leq \|x\|^2 \|y\|^2$.

Corollary (Triangle inequality)

For the Euclidean norm, $\|x + y\| \leq \|x\| + \|y\|$.

Interior, open, closed, closure, boundary

- x_0 is **interior** to A if some ball $B(x_0, \varepsilon) \subseteq A$.
- A is **open** if $A = \mathrm{int}(A)$; it is **closed** if A^c is open.
- **Closure** $\bar{A}$: the smallest closed set containing A.
- **Boundary** $\partial A = \bar{A} \setminus \mathrm{int}(A)$.

Open and closed are not opposites: $\mathbb{R}^n$ and $\emptyset$ are both open and closed, while $[0, 1)$ is neither.

Topology: the facts we use (stated)

- Arbitrary unions of open sets are open; finite intersections of open sets are open.
- Arbitrary intersections of closed sets are closed; finite unions of closed sets are closed.

Only finite on the intersection side: $\bigcap_n \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$ is closed.

Why closedness matters: tomorrow, closed and bounded gives compact in $\mathbb{R}^n$, and a continuous function on a compact set attains its maximum. Closedness is what makes an optimal choice exist.

Sequences and convergence

Definition

A **sequence** in X is a function $x : \mathbb{N} \to X$. Writing $x_k := x(k)$, we denote it $\{x_k\}_{k=1}^{\infty}$ (for us $X = \mathbb{R}^n$, or a function space).

Definition

$\{x_k\} \subseteq X$ **converges** to x^* if for every $\varepsilon > 0$ there is an N with $d(x_k, x^*) < \varepsilon$ for all $k \geq N$.

Uniqueness of the limit. If $x_k \to x$ and $x_k \to y$, then $x = y$.

Closed sets, seen through sequences

Theorem

A is closed if and only if every convergent sequence contained in A has its limit in A.

This is the bridge between the topology and the sequences, and it returns all week.

Cauchy sequences and completeness

Definition

$\{x_k\}$ is **Cauchy** if for every $\varepsilon > 0$ there is an N with $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

Definition

(X, d) is **complete** if every Cauchy sequence converges to a point of X.

Convergent always implies Cauchy; the converse is exactly what completeness buys.

Complete spaces, and why we care

- $\mathbb{R}^n$ with the Euclidean metric is **complete**.
- $\mathbb{Q}$ is **not** complete: rationals increasing to $\sqrt{2}$ are Cauchy, but the limit is not in $\mathbb{Q}$.

Completeness is the engine of the next lecture: a complete space plus a contraction gives the Banach fixed point theorem, hence existence and uniqueness of value functions on Wednesday.

References

- A. de la Fuente, *Mathematical Methods and Models for Economists*, Ch. 1 and 2 (Sections 2.1 to 2.3).
- E. A. Ok, *Real Analysis with Economic Applications*, Ch. A and B.
- C. P. Simon and L. Blume, *Mathematics for Economists*, Ch. 12 and 13.
- B. P. Rynne and M. A. Youngson, *Linear Functional Analysis*, Ch. 3 (inner products).