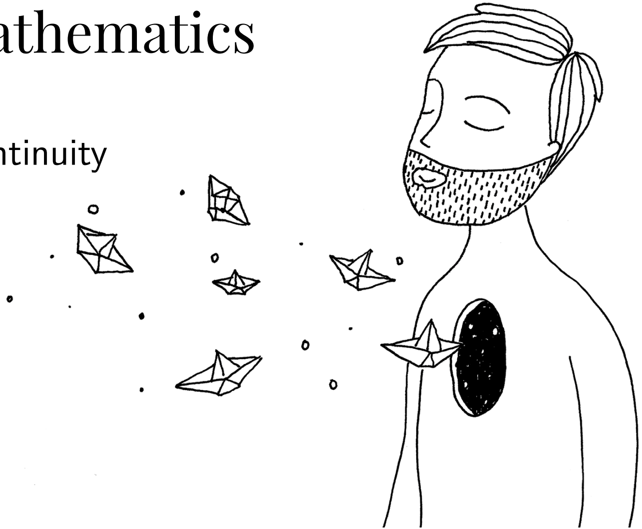
# 4509 - Bridging Mathematics

## Lecture 2: Compactness, Continuity and the Banach Fixed Point Theorem



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# Objectives

- Understand compactness in general metric spaces (open-cover definition) and in $\mathbb{R}^n$ (Heine-Borel, Bolzano-Weierstrass).
- Master continuous functions through their sequential and topological (preimage) characterizations.
- State Weierstrass's Extreme Value Theorem and prove the Banach Contraction Mapping Theorem.

Two existence machines:

- *compact* $+$ *continuous* $\Rightarrow$ *a maximum exists*;
- *complete* $+$ *contraction* $\Rightarrow$ *a unique fixed point exists and can be computed by iterating*.

# Compactness: two definitions

## Definition

$K$ is **compact** if every open cover of $K$ has a finite subcover:

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha \ (U_\alpha \text{ open}) \implies \exists\, \alpha_1, \ldots, \alpha_m : \ K \subseteq \bigcup_{i=1}^{m} U_{\alpha_i}.$$

## Definition

$K$ is **sequentially compact** if every sequence in $K$ has a subsequence converging to a point of $K$.

In a metric space the two coincide. Economists almost always use the sequential one. Compactness is a finiteness property in disguise: it lets you pass from "for each point..." to "uniformly."

# Compactness in $\mathbb{R}^n$

## Theorem (Heine-Borel)

*$K \subseteq \mathbb{R}^n$ is compact if and only if $K$ is **closed and bounded**.*

## Theorem (Bolzano-Weierstrass)

*Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

In $\mathbb{R}^n$ you never check covers: you check "closed and bounded." Both rest on one bisection: a bounded sequence lives in a box; halve the box, keep the half with infinitely many terms, repeat; the nested halves shrink to a point, and closedness catches that limit inside the set.

# The warning: Heine-Borel fails in infinite dimensions

In $C[0,1]$ with the sup norm, the **closed unit ball is not compact**. Take continuous tent functions of height $1$ on the disjoint intervals $[\frac{1}{n+1}, \frac{1}{n}]$, with midpoints $c_n = \frac{2n+1}{2n(n+1)}$:

$$f_n(x) = \begin{cases} 2n(n+1)\,x - 2n, & x \in [\frac{1}{n+1},\, c_n], \\ 2(n+1) - 2n(n+1)\,x, & x \in [c_n,\, \frac{1}{n}], \\ 0, & \text{otherwise.} \end{cases}$$

Each has $\|f_n\|_\infty = 1$, and disjoint supports give $d_\infty(f_n, f_m) = 1$ for $n \neq m$: no convergent subsequence.

Value functions, price paths and consumption streams live in infinite-dimensional spaces, where "closed and bounded" no longer buys compactness. This is why macroeconomists reach for the contraction route of Part 3 rather than "compactness + Weierstrass."

# Continuity: three equivalent definitions

For $f : X \to Y$ between metric spaces, the following are equivalent:

1. **Topological:** $f^{-1}(V)$ is open in $X$ for every open $V \subseteq Y$.
2. **Sequential:** $x_k \to x_0 \;\Rightarrow\; f(x_k) \to f(x_0)$.
3. **$\varepsilon$-$\delta$**: $\forall \varepsilon > 0 \; \exists \delta > 0 : \; d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon$.

Definition 1 is the deep one: "pulling back open sets keeps them open." It reappears verbatim for correspondences on Tuesday (upper/lower hemicontinuity are just this, split in two).

# Weierstrass Extreme Value Theorem

## Theorem

*The continuous image of a compact set is compact.*

## Theorem (Weierstrass)

*If $K \subseteq \mathbb{R}^n$ is nonempty and compact and $f : K \to \mathbb{R}$ is continuous, then $f$ attains a global max and min:*

$$\exists\, x_{\min}, x_{\max} \in K : \; f(x_{\min}) \leq f(x) \leq f(x_{\max}) \;\; \forall x \in K.$$

Idea: $f(K)$ is compact in $\mathbb{R}$, hence closed and bounded; bounded gives a supremum, closed makes it achieved. This is why Lecture 1's "closed" mattered.

# Why Weierstrass matters

Weierstrass is the licence to write "$\arg\max$."

A consumer chooses from a compact budget set; a continuous utility guarantees an optimum **exists**. Drop compactness (an open budget set, unbounded choices) and the demand may fail to exist. Every $\max$ in micro leans on this.

# Contractions

## Definition

On a metric space $(X, d)$, a map $T : X \to X$ is a **contraction** with modulus $\beta \in [0, 1)$ if

$$d(Tx, Ty) \leq \beta \, d(x, y) \qquad \forall x, y.$$

$T$ shrinks all distances by at least a factor $\beta < 1$.

The strictness matters: $\beta = 1$ (an isometry, or $x \mapsto x + 1$) is *not* a contraction and can have no fixed point.

# The Banach Contraction Mapping Theorem

## Theorem (Banach)

*If $(X, d)$ is **complete** and $T$ is a contraction with modulus $\beta < 1$, then:*

1. *$T$ has a **unique** fixed point $x^*$ ($Tx^* = x^*$);*
2. *from **any** $x_0$, the iteration $x_{k+1} = Tx_k$ converges to $x^*$;*
3. *error bound:* $d(x_k, x^*) \le \dfrac{\beta^k}{1-\beta}\, d(x_1, x_0)$.

This is the keystone of the week. It returns on Wednesday as the reason a Bellman value function exists, so we prove it in full.

# Why Banach is the keystone

$$\underbrace{\text{complete space}}_{\text{L01}} + \underbrace{\text{contraction}}_{\text{today}} \Rightarrow \exists! \text{ fixed point } \xrightarrow{\text{L08}} \underbrace{TV = V,\ V = \text{value function}}_{\text{Bellman}}.$$

- **Dynamic programming:** the Bellman operator $TV = \max\{F + \beta V\}$ is a contraction in the sup norm; Banach gives a unique value function, and value-function iteration converges to it.
- **ODEs (Picard-Lindelöf):** existence and uniqueness of the solution to $\dot{x} = f(x)$ is a contraction argument, the same theorem.
- **Markets/algorithms:** *tâtonnement*, Gauss-Seidel and many equilibrium solvers converge because the update map is a contraction.

The error bound is not decoration: it tells a computational economist how many iterations reach a tolerance: geometric convergence at rate $\beta$, the discount factor.

# A worked contraction

Solve $x = \frac{1}{2}x + 3$ by iteration. Here $Tx = \frac{1}{2}x + 3$ is a contraction with $\beta = \frac{1}{2}$.

From $x_0 = 0$: $3,\ 4.5,\ 5.25,\ 5.625,\ \ldots\ \to\ 6 = x^*$, and indeed $T6 = 6$.

A linear steady state $x^* = \frac{b}{1-a}$ is a fixed point, and $|a| < 1$ (stability) is exactly the contraction condition. This ties forward to stability of dynamic systems and to the consumption rule $c_0 = (1 - \beta)x_0$ later this week.

# References


- A. de la Fuente, *Mathematical Methods and Models for Economists*, Ch. 2 (Sections 2.4 to 2.7).
- E. A. Ok, *Real Analysis with Economic Applications*, Ch. B and C.
- R. K. Sundaram, *A First Course in Optimization Theory*, Ch. 1 and 9.