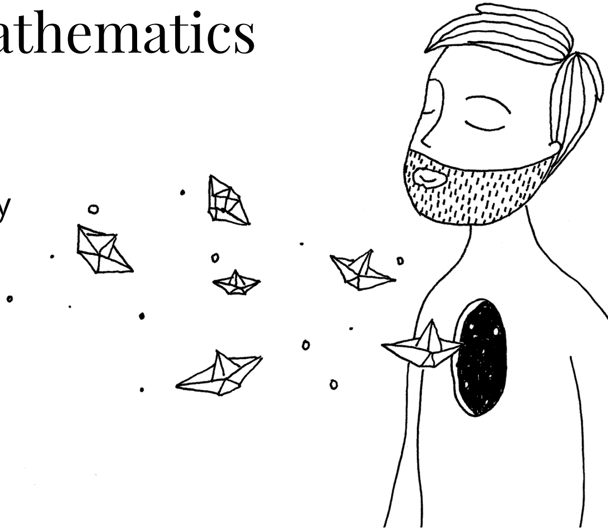
4509 - Bridging Mathematics

Lecture 4: Convex Sets, Concavity and Quasiconcavity



PAULO FAGANDINI

NOVA SCHOOL OF BUSINESS & ECONOMICS

Objectives

- Formalize convex sets, convex combinations and the **separating and supporting hyperplane theorems**, the analytical basis of competitive pricing.
- Characterize concave and convex functions through the C^1 tangent inequality and the C^2 Hessian, cashing the curvature bridge from Lecture 3.
- Master **quasiconcavity**, the ordinal cousin of concavity that consumer and producer theory are actually built on.

The spine: convexity gives you **prices**. Two disjoint convex sets can always be separated by a hyperplane, and its normal is a price vector, the welfare theorems in one picture. Quasiconcavity is the property of preferences that survives relabelling utility.

Convex sets

Definition

$C \subseteq \mathbb{R}^n$ is **convex** if for all $x, y \in C$ and $t \in [0, 1]$,

$$tx + (1 - t)y \in C.$$

The whole segment joining any two points stays inside. Intersections of convex sets are convex.

Say it and write it: a **concave set is not a thing**. Only *functions* are concave or convex; *sets* are convex or not.

Hyperplanes and separation

A **hyperplane** $H(p, c) = \{x : p^\top x = c\}$ with normal $p \neq 0$ (read p as a **price vector**), splitting $\mathbb{R}^n$ into half-spaces $\{p^\top x \gtrless c\}$. The line $p^\top x = c$ is a budget line.

Theorem (Separating hyperplane, Minkowski)

C_1, C_2 nonempty, disjoint, convex $\Rightarrow$ there exist $p \neq 0$ and c with $p^\top x \leq c \leq p^\top y$ for all $x \in C_1$, $y \in C_2$.

Theorem (Supporting hyperplane)

C closed convex, $x_0 \in \partial C \Rightarrow$ there is $p \neq 0$ with $p^\top x \leq p^\top x_0$ for all $x \in C$.

You can always slide a flat wall between two convex bodies that do not touch. The separating direction is the line of shortest distance between the sets, which you find by projection (Lecture 3).

Why separation is welfare economics

- **Second Welfare Theorem:** the “better-than” set (convex, by convex preferences) and the aggregate production set (convex) are disjoint at a Pareto optimum; the separating normal p is a **price vector** that decentralizes it as a competitive equilibrium. Prices *are* the normal to the separating hyperplane.
- **No-arbitrage:** attainable payoffs and the positive orthant are separated by a vector of **state prices**, the Fundamental Theorem of Asset Pricing.
- Farkas, LP duality and minimax are all separation in disguise.

Whenever an economist pulls a price out of thin air to support an allocation, it is this theorem.

Concave and convex functions

Definition

On a convex set C, f is **concave** if for all $x, y \in C$, $t \in [0,1]$,

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y),$$

convex if the inequality reverses, and **strict** if strict for $x \neq y$, $t \in (0,1)$.

Concave means the chord lies below the graph: the function bulges up, diminishing returns.

This ties back to convex sets: f is concave if and only if its **hypograph** $\{(x,y) : y \leq f(x)\}$ is a convex set. A concave function is a convex region underneath.

Derivative characterizations

Let C be open convex and $f \in C^1$.

Theorem (Tangent, C^1)

f concave $\iff f(y) \leq f(x) + \nabla f(x)^\top (y - x)$ for all $x, y \in C$.

The graph lies below every tangent plane: for a concave function the tangent always overpredicts.

Theorem (Curvature, C^2)

*f concave $\iff$ the Hessian $H(f, x)$ is negative semidefinite for all x; $H \prec 0$ everywhere $\Rightarrow$ **strictly** concave.*

Recall from Lecture 3: concavity is negative semidefiniteness of the Hessian, everywhere.

Why concavity is the workhorse assumption

Theorem

*If f is concave and $\nabla f(x^*) = 0$, then x^* is a **global** maximizer.*

Direct from the tangent inequality: $f(y) \leq f(x^*) + \nabla f(x^*)^\top (y - x^*) = f(x^*)$ for all y.

For a concave problem, "set the derivative to zero" is sufficient for a **global optimum** (in the interior of a set). That is why producer theory (concave profit) and much of consumer theory requires this.

Quasiconcavity: why we need it

Utility is **ordinal**: if u represents preferences, so does $g \circ u$ for any strictly increasing g. But **concavity is destroyed by relabelling**: take u concave, apply a convex g, and $g \circ u$ need not be concave, though preferences have not changed.

We need a property of the preference *ordering*, not of the arbitrary numbers attached to it. That property is **quasiconcavity**.

Quasiconcavity: definitions

Definition

The **upper contour set** at level a is $U_a = \{x \in C : f(x) \geq a\}$, the "at least as good as a" set. f is **quasiconcave** if every U_a is convex.

Theorem (Equivalent algebraic form)

f quasiconcave $\iff$ $f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$ *for all* $x, y \in C$, $t \in [0,1]$.

Quasiconcave means convex better-than sets, that is, indifference curves bowed to the origin, that is, a diminishing marginal rate of substitution. The whole shape of an indifference map is a quasiconcavity statement.

Quasiconcavity is ordinally invariant

Theorem

If f is quasiconcave and g is strictly increasing, then $g \circ f$ is quasiconcave.

Proof. The upper contour set of $g \circ f$ at level a is

$$\{x : g(f(x)) \geq a\} = \{x : f(x) \geq g^{-1}(a)\} = U_{g^{-1}(a)}(f),$$

which is convex because f is quasiconcave. Every upper contour set of $g \circ f$ is a relabelled upper contour set of f, so $g \circ f$ is quasiconcave. ■

The sets do not move when you relabel, only their labels do. That is exactly the invariance we wanted, and concavity does not have it.

Bordered Hessian and a warning example

C^2 **test.** With $\bar{B}(x) = \begin{pmatrix} 0 & \nabla f^\top \\ \nabla f & H \end{pmatrix}$, quasiconcavity corresponds to an alternating-sign pattern of the leading principal minors. It tests curvature only along moves that keep you on a level set ($\nabla f^\top h = 0$): curvature *along* the indifference curve, not in every direction.

Cobb-Douglas. $u(x_1, x_2) = x_1^\alpha x_2^\beta$, $\alpha, \beta > 0$:

- concave only if $\alpha + \beta \leq 1$ (decreasing returns);
- **always** quasiconcave: the sets $\{x_1^\alpha x_2^\beta \geq a\}$ are convex.

Remark: demand theory requires quasiconcavity, not concavity. A Cobb-Douglas consumer with increasing returns ($\alpha + \beta > 1$) is not concave, yet its upper contour sets stay convex, so demand is still well-behaved.

Why the block hangs together

Convexity of preferences and technology delivers the whole week:

- separation $\Rightarrow$ **prices** support optima (welfare theorems);
- quasiconcavity $\Rightarrow$ **convex** demand and best-response sets $\Rightarrow$ the correspondences of Lecture 5 are convex-valued $\Rightarrow$ **Kakutani applies** $\Rightarrow$ equilibrium exists (Lecture 6).

The convexity assumed here is what delivers existence across microeconomics.

References

- R. K. Sundaram, *A First Course in Optimization Theory*, Ch. 7 and 8.
- A. de la Fuente, *Mathematical Methods and Models for Economists*, Ch. 4.
- C. P. Simon and L. Blume, *Mathematics for Economists*, Ch. 21.