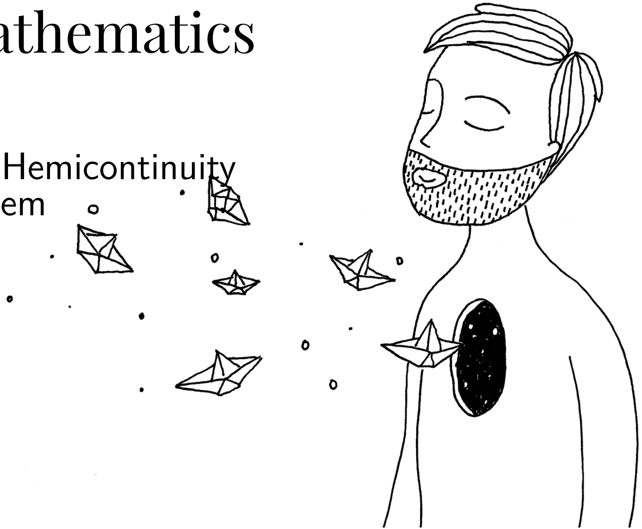
4509 - Bridging Mathematics

Lecture 5: Correspondences, Hemicontinuity and Berge's Maximum Theorem



PAULO FAGANDINI

Objectives

- Define a set-valued mapping (correspondence) $\Gamma : X \rightrightarrows Y$ and its role in microeconomics: budget sets, demand, best responses.
- Characterize **upper** and **lower hemicontinuity** and the closed graph theorem.
- Master **Berge's Maximum Theorem**: continuity of the value function and upper hemicontinuity of the argmax.

The spine: when the optimizer is a set, continuity splits in two, the set must not suddenly **explode** (UHC) nor suddenly **collapse** (LHC). Berge then guarantees that maximizing a continuous objective over a continuously-moving constraint yields a continuous value function and an upper-hemicontinuous argmax. This is what legitimizes comparative statics.

Correspondences

Definition

A **correspondence** $\Gamma : X \rightrightarrows Y$ assigns to each x a nonempty subset $\Gamma(x) \subseteq Y$. A function is the single-valued special case.

- Budget set $B(p, w) = \{x \geq 0 : p^\top x \leq w\}$.
- Walrasian demand $x^*(p, w) = \arg\max_{x \in B(p,w)} u(x)$, **multi-valued** if u is not strictly quasiconcave (a flat indifference segment gives a whole segment of optima).
- Best response $BR_i(s_{-i}) = \arg\max_{s_i} u_i(s_i, s_{-i})$, whose fixed point is Nash (Lecture 6).
- Feasible transitions $\Gamma(x_t)$ in dynamic programming (Lecture 8).

<u>Remark:</u> whenever choices are not unique, or the feasible set moves with a parameter, the object is a correspondence, not a function.

Graph and closed graph

Definition

The **graph** is $\mathrm{Gr}(\Gamma) = \{(x, y) : y \in \Gamma(x)\}$. Γ has a **closed graph** if $\mathrm{Gr}(\Gamma)$ is a closed set:

$$x_k \to x, \quad y_k \to y, \quad y_k \in \Gamma(x_k) \quad \Rightarrow \quad y \in \Gamma(x).$$

The closed graph is the set-valued reading of "no sudden jumps", and the property Kakutani requires (Lecture 6).

Hemicontinuity: the definitions

Definition (UHC at x_0)

For every open $V \supseteq \Gamma(x_0)$ there is a neighbourhood U of x_0 with $\Gamma(x) \subseteq V$ for all $x \in U$.

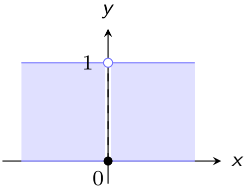
Definition (LHC at x_0)

For every open V with $\Gamma(x_0) \cap V \neq \varnothing$ there is a neighbourhood U of x_0 with $\Gamma(x) \cap V \neq \varnothing$ for all $x \in U$.

Γ is **continuous** if it is both UHC and LHC. Informally, UHC keeps the whole set inside any neighbourhood of $\Gamma(x_0)$; LHC keeps every point of $\Gamma(x_0)$ approximated by nearby sets.

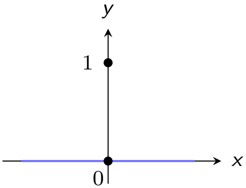
Test yourself: which fails UHC, which fails LHC?

(A)



$\Gamma(x) = [0, 1]$ for $x \neq 0$, $\Gamma(0) = \{0\}$

(B)



$\Gamma(x) = \{0\}$ for $x \neq 0$, $\Gamma(0) = \{0, 1\}$

More practice: UHC and LHC?

For $\Gamma : [0,1] \rightrightarrows [0,1]$, classify:

1. $\Gamma(x) = (a, b) \subset [0,1]$ (constant open interval);
2. $\Gamma(x) = (0.2 + 0.2x,\ 0.4 + 0.2x)$ (moving open interval);
3. $\Gamma(x) = [0.2 + 0.2x,\ 0.4 + 0.2x]$ (moving closed interval).

Sequential characterizations (compact-valued)

The forms used in practice:

- **UHC at** $x_0 \iff$ for all $x_k \to x_0$ and $y_k \in \Gamma(x_k)$, some subsequence $y_{k_j} \to y_0 \in \Gamma(x_0)$.
- **LHC at** $x_0 \iff$ for all $x_k \to x_0$ and every $y_0 \in \Gamma(x_0)$, there exist $y_k \in \Gamma(x_k)$ with $y_k \to y_0$.

Theorem (Closed graph $\Rightarrow$ UHC, compact-valued)

Take $x_k \to x_0$, $y_k \in \Gamma(x_k)$. Values lie in a compact set, so by Bolzano-Weierstrass (Lecture 2) a subsequence $y_{k_j} \to y_0$. Then $(x_{k_j}, y_{k_j}) \to (x_0, y_0)$ in the closed graph, so $y_0 \in \Gamma(x_0)$: sequential UHC. ■

Compactness yields a convergent subsequence; the closed graph places its limit inside the set. Same argument as Lecture 1, in the product space.

Berge's Maximum Theorem

Theorem

For $V(\theta) = \max_{x \in \Gamma(\theta)} f(x, \theta)$ *and* $x^*(\theta) = \arg\max_{x \in \Gamma(\theta)} f(x, \theta)$*, if* f *is* ***continuous*** *and* Γ *is* ***compact-valued*** *and* ***continuous*** *(UHC and LHC), then*

1. V *is continuous;*
2. x^* *is nonempty, compact-valued and UHC;*
3. *if* $x^*(\theta)$ *is single-valued, it is a continuous function.*

Role of each hypothesis:

- compact values + continuous $f \Rightarrow$ the max *exists* (Weierstrass, Lecture 2), so x^* is nonempty;
- LHC prevents V from jumping *down* (near-optimal points survive nearby);
- UHC prevents V from jumping *up* and the argmax from exploding.

The two one-sided bounds force V continuous. The argmax may still be multi-valued, hence a UHC correspondence rather than a function, unless the optimum is unique.

Application: the consumer

Set $\theta = (p, w)$, $\Gamma(p, w) = B(p, w)$, objective $u(x)$.

- $B(p, w)$ is compact (for $p \gg 0$, $w > 0$) and continuous in (p, w).
- u is continuous.

Berge gives: indirect utility $v(p, w)$ is **continuous**, and Walrasian demand $x^*(p, w)$ is **UHC**. If u is strictly quasiconcave (Lecture 4), the optimum is unique and demand is a **continuous function**.

Berge is what licenses comparative statics: demand and welfare respond continuously to income and prices. It also supplies the inputs Kakutani needs (Lecture 6) and the upper hemicontinuity of the DP policy correspondence (Lecture 8).

References

- E. A. Ok, *Real Analysis with Economic Applications*, Ch. E.
- R. K. Sundaram, *A First Course in Optimization Theory*, Ch. 9.
- A. de la Fuente, *Mathematical Methods and Models for Economists*, Ch. 2 (Section 2.7).