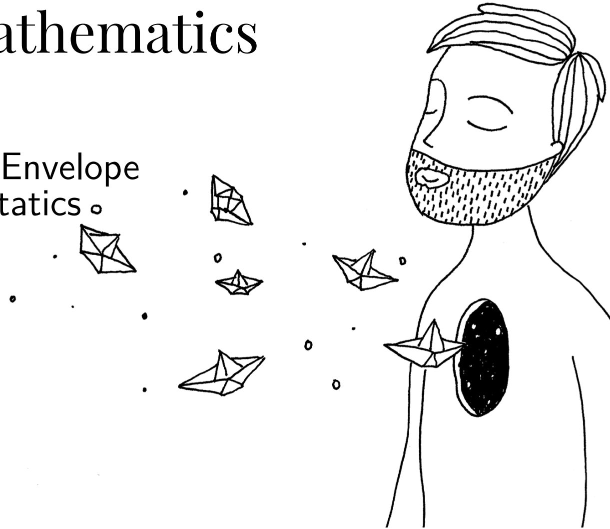
4509 - Bridging Mathematics

Lecture 6: Fixed Points, the Envelope Theorem and Comparative Statics



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Objectives

- Understand fixed point theorems for functions (**Brouwer**) and correspondences (**Kakutani**), the engine behind existence of Walrasian and Nash equilibrium.
- Master the **Envelope Theorem** across unconstrained and constrained problems, and recover Roy, Shephard and Hotelling from it.
- Preview monotone comparative statics and supermodularity.

The spine: two questions about optima. **Existence**, does an equilibrium exist, answered by fixed points. **Sensitivity**, how does the *value* of an optimum move with a parameter, answered by the envelope theorem, whose punchline is that at an optimum only the direct effect matters.

Brouwer's fixed point theorem

Theorem (Brouwer)

*$K \subseteq \mathbb{R}^n$ nonempty, **compact** and **convex**, $f : K \to K$ continuous $\Rightarrow$ there exists x^* with $f(x^*) = x^*$.*

A continuous map of an interval into itself cannot avoid its own diagonal. In higher dimension the reason is topological: there is no continuous retraction of a ball onto its boundary, and a fixed-point-free map would build one.

Every hypothesis matters

Dropping any one breaks the theorem:

- **Not closed:** $f(x) = x/2$ on $(0,1)$. The fixed point would be $0 \notin (0,1)$.
- **Not bounded:** $f(x) = x + 1$ on $[0,\infty)$. Pushes off to infinity.
- **Not convex:** rotation by π of the circle S^1. Every point moves.

<u>Remark:</u> verify compactness and convexity before invoking Brouwer.

Kakutani: the set-valued upgrade

Theorem (Kakutani)

*K nonempty compact convex, $\Gamma : K \rightrightarrows K$ with (i) $\Gamma(x)$ nonempty and **convex** for all x, (ii) **closed graph** (equivalently UHC compact-valued, Lecture 5). Then there exists $x^* \in \Gamma(x^*)$.*

Brouwer for correspondences: convex values patch the single-valued proof, the closed graph plays the role of continuity. Approximate Γ by a continuous function (one point over each grid vertex, interpolated; convex values keep it inside Γ), apply Brouwer, refine the grid, and the closed graph catches the limit inside $\Gamma(x^*)$.

Fixed points are the existence engine

- **Nash equilibrium:** the strategy simplex $S = \prod_i S_i$ is compact convex; the best-response correspondence $BR : S \rightrightarrows S$ is convex-valued (quasiconcave payoffs, Lecture 4) and UHC (Berge, Lecture 5), so **Kakutani** gives $s^* \in BR(s^*)$, a Nash equilibrium. Precisely the construction of Lectures 4 and 5.
- **Walrasian equilibrium:** the excess-demand correspondence on the price simplex has a fixed point, so market-clearing prices exist.
- **Bellman value function (Lecture 8):** a fixed point of the Bellman operator.

"An equilibrium exists" almost always means "a certain map has a fixed point."

The Envelope Theorem: unconstrained

$V(\theta) = \max_x f(x,\theta)$, optimizer $x^*(\theta)$, interior FOC $\nabla_x f(x^*,\theta) = 0$. Differentiate the identity $V(\theta) = f(x^*(\theta),\theta)$ by the chain rule:

$$\frac{dV}{d\theta} = \underbrace{\nabla_x f(x^*,\theta)^\top}_{=0 \text{ (FOC)}} \frac{dx^*}{d\theta} + \frac{\partial f}{\partial \theta} = \boxed{\frac{\partial f(x^*(\theta),\theta)}{\partial \theta}.}$$

At an optimum the **indirect** effect, through re-optimizing x, vanishes: a small re-adjustment is first-order free. Only the **direct** effect of θ survives.

The Envelope Theorem: constrained

$V(\theta) = \max_x f(x, \theta)$ subject to $g_j(x, \theta) \leq 0$, Lagrangian $\mathcal{L} = f - \sum_j \lambda_j g_j$. Then

$$\frac{\partial V}{\partial \theta_k} = \frac{\partial \mathcal{L}}{\partial \theta_k}\bigg|_{x^*, \lambda^*} = \frac{\partial f}{\partial \theta_k} - \sum_j \lambda_j^* \frac{\partial g_j}{\partial \theta_k}.$$

Multiplier = shadow price. If θ_k relaxes constraint j (say more income), then $\partial V / \partial \theta_k = \lambda_j^*$, the marginal value of relaxing it: λ_j^* is the price of one more unit of the constraint. It is also the normal of the supporting hyperplane at the optimum (Lecture 4).

The three classic identities

Let $v(p, w) = \max\{u(x) : p^\top x \leq w\}$, multiplier λ^* (marginal utility of income).

- **Roy's identity.** Envelope on p_i and on w: $\dfrac{\partial v}{\partial p_i} = -\lambda^* x_i^*$ and $\dfrac{\partial v}{\partial w} = \lambda^*$, so

$$\boxed{x_i^*(p, w) = -\frac{\partial v/\partial p_i}{\partial v/\partial w}.}$$

  Demand recovered from indirect utility, no re-optimization.
- **Shephard's lemma.** $e(p, u) = \min\{p^\top x : u(x) \geq u\} \Rightarrow \partial e/\partial p_i = h_i^*(p, u)$ (Hicksian demand).
- **Hotelling's lemma.** $\pi(p) = \max\{py - c(y)\} \Rightarrow \partial \pi/\partial p = y^*(p)$ (supply).

Same theorem three times: derivatives of value functions are the optimal quantities. The value function is differentiated, never re-solved. This is the backbone of duality in micro.

Monotone comparative statics (preview)

When concavity or differentiability fail (discrete choice, non-smooth), signing $dx^*/d\theta$ still works through **order** structure. If $f(x, \theta)$ has **increasing differences** in (x, θ) (supermodularity), then $x^*(\theta)$ is nondecreasing, with no derivatives (Topkis, Milgrom-Shannon).

Raising θ raises the *marginal* return to x, so the optimal x can only rise. Pure order logic, which survives where the envelope and FOC machinery cannot go.

References

- R. K. Sundaram, *A First Course in Optimization Theory*, Ch. 9 and 10.
- E. A. Ok, *Real Analysis with Economic Applications*, Ch. C (Section C.5) and Ch. E.
- A. de la Fuente, *Mathematical Methods and Models for Economists*, Ch. 8.