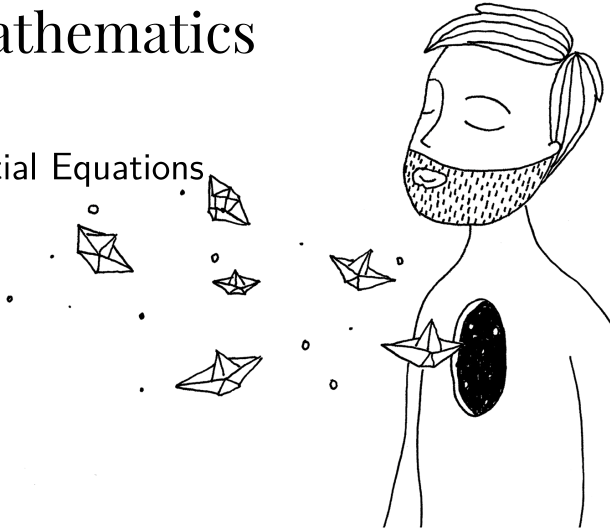
4509 - Bridging Mathematics

Lecture 7: Ordinary Differential Equations and 2D Phase Diagrams



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Objectives

- Solve linear systems $\dot{x} = Ax$ with the matrix exponential and the eigenvalue decomposition.
- Classify 2D phase portraits (saddles, nodes, spirals, centers) from the trace-determinant plane.
- Analyze nonlinear autonomous systems: phase lines, arrows, and linearization through the Jacobian.

Scalar warm-up

$$\dot{x} = ax + b \implies x(t) = \left(x(0) + \frac{b}{a}\right) e^{at} - \frac{b}{a}, \qquad x^* = -\frac{b}{a}$$

Stable if and only if $a < 0$.

The matrix exponential e^{At}

Proposition

The function $x(t) = e^{At}x(0)$ *solves* $\dot{x} = Ax$.

If $Av = \lambda v$ then $A^k v = \lambda^k v$, so

$$e^{At}v = \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda^k v = e^{\lambda t}v.$$

The eigenvalue solution

Expand $x(0) = \sum_i c_i v_i$; by linearity

$$\boxed{x(t) = e^{At} x(0) = \sum_i c_i \, e^{\lambda_i t} \, v_i.}$$

- Each mode grows like $e^{\lambda_i t}$.
- Explodes if some $\operatorname{Re} \lambda_i > 0$; decays to 0 if all $\operatorname{Re} \lambda_i < 0$.
- Complex $\lambda = \alpha \pm i\beta$ gives $e^{\alpha t}(\cos \beta t \pm i \sin \beta t)$: a spiral at frequency β, amplitude set by α.
- Repeated eigenvalues add polynomial factors $t \, e^{\lambda t}$; $\operatorname{Re} \lambda$ still decides stability.

Worked example: solving $\dot{x} = Ax$ (1/2)

Take

$$\dot{x} = Ax, \qquad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Eigenvalues. Characteristic polynomial

$$\det(A - \lambda I) = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0 \;\Rightarrow\; \lambda_1 = 3,\; \lambda_2 = -1.$$

Eigenvectors. Solve $(A - \lambda_i I)v_i = 0$:

$$\lambda_1 = 3: \; \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} v_1 = 0 \;\Rightarrow\; v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\lambda_2 = -1: \; \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} v_2 = 0 \;\Rightarrow\; v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Worked example: solving $\dot{x} = Ax$ (2/2)

By the modal solution,

$$x(t) = c_1\, e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\, e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Fit the initial condition $x(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$: from $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$,

$$c_1 = \tfrac{x_0 + y_0}{2}, \qquad c_2 = \tfrac{x_0 - y_0}{2}.$$

Read the dynamics. $\lambda_1 = 3 > 0 > \lambda_2 = -1$: a **saddle**. The stable manifold is $\mathrm{span}\{(1,-1)\}$ (the e^{-t} mode); every trajectory with $c_1 \neq 0$ explodes along $(1,1)$. Only initial states with $x_0 = -y_0$ (so $c_1 = 0$) converge to the origin.

The trace-determinant plane

For $A \in \mathbb{R}^{2\times 2}$: $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0$, discriminant $\Delta = \operatorname{tr}(A)^2 - 4\det(A)$.

- $\det < 0 \;\Rightarrow$ saddle (real λ of opposite sign).
- $\det > 0,\ \operatorname{tr} < 0,\ \Delta \geq 0 \;\Rightarrow$ stable node (sink).
- $\det > 0,\ \operatorname{tr} > 0,\ \Delta \geq 0 \;\Rightarrow$ unstable node (source).
- $\Delta < 0 \;\Rightarrow$ spiral, stable if $\operatorname{tr} < 0$; $\operatorname{tr} = 0$ gives a center.

The saddle

One stable eigenvalue $\lambda_1 < 0$ with a 1D stable manifold (the saddle path) and one unstable $\lambda_2 > 0$.

With one predetermined (state) variable and one jump (control) variable, the control must jump onto the saddle path: exactly one choice avoids explosion. This is determinacy, a unique non-explosive equilibrium path.

Nonlinear systems: linearization

For autonomous $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, a steady state $(\bar{x}, \bar{y})$ solves $f = g = 0$. Linearize with the Jacobian

$$J(\bar{x}, \bar{y}) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}_{(\bar{x},\bar{y})}, \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx J \begin{pmatrix} x - \bar{x} \\ y - \bar{y} \end{pmatrix}.$$

Theorem (Hartman-Grobman)

If J has no eigenvalue with zero real part (hyperbolic), the nonlinear system near $(\bar{x}, \bar{y})$ looks topologically like its linearization.

The phase-diagram recipe

1. Phase lines. Draw $\dot{x} = 0$ ($f = 0$) and $\dot{y} = 0$ ($g = 0$); their intersection is the steady state.
2. Sign the regions. A test point in each region gives the sign of $\dot{x}$ (horizontal arrow) and of $\dot{y}$ (vertical arrow).
3. Compose the horizontal and vertical arrows into a resultant per region.
4. Trace manifolds: the stable (saddle-path) and unstable trajectories through the steady state.

Worked example: a nonlinear phase diagram (1/4)

Consider the autonomous system

$$\dot{x} = f(x, y) = y - x, \qquad \dot{y} = g(x, y) = x - y^2.$$

Phase lines (set each rate to zero):

$$\dot{x} = 0 \;\Rightarrow\; y = x, \qquad\qquad \dot{y} = 0 \;\Rightarrow\; x = y^2.$$

Steady states are their intersections: substituting $y = x$ into $x = y^2$ gives $x = x^2$, so $x \in \{0, 1\}$:

$$(\bar{x}, \bar{y}) \in \{(0, 0),\ (1, 1)\}.$$

Worked example: linearize and classify (2/4)

The Jacobian is

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -2y \end{pmatrix}.$$

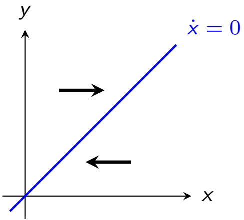
At $(0,0)$**:** $J = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$, $\ \mathrm{tr} = -1,\ \det = -1 < 0$, so $\lambda = \frac{-1 \pm \sqrt{5}}{2}$ (opposite signs): a **saddle**.

At $(1,1)$**:** $J = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$, $\ \mathrm{tr} = -3,\ \det = 1,\ \Delta = 5 > 0$, so $\lambda = \frac{-3 \pm \sqrt{5}}{2} < 0$: a **stable node**.

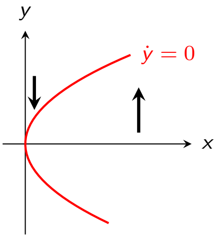
Both Jacobians are hyperbolic, so by Hartman-Grobman these local verdicts carry over to the nonlinear system.

Worked example: sign the regions (3/4)

Each phase line splits the plane in two; read off the sign of the *rate* on each side.

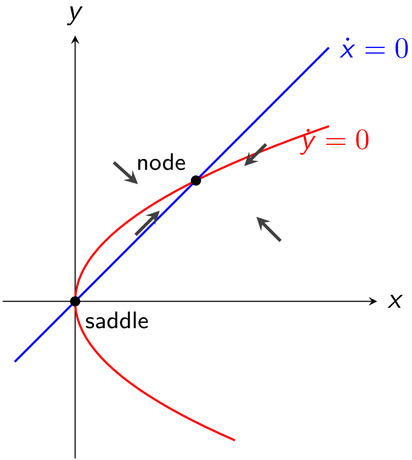


$\dot{x} = y - x$. **Above** the line ($y > x$) $\dot{x} > 0$, so x grows: arrow $\rightarrow$. **Below** ($y < x$) $\dot{x} < 0$: arrow $\leftarrow$.



$\dot{y} = x - y^2$. To the **right** of the parabola ($x > y^2$) $\dot{y} > 0$, so y grows: arrow $\uparrow$; to the **left** ($x < y^2$): arrow $\downarrow$. For a sideways parabola say right/left, not above/below.

Worked example: the phase portrait (4/4)



In each region, compose the horizontal arrow (from $\dot{x}$) with the vertical one (from $\dot{y}$):

- $y > x,\ x > y^2$: $\nearrow$
- $y > x,\ x < y^2$: $\searrow$
- $y < x,\ x > y^2$: $\nwarrow$
- $y < x,\ x < y^2$: $\swarrow$

The composed flow spirals into the **stable node** $(1, 1)$; the **saddle** at the origin has one stable direction separating the basins.

Enrichment: Markov chains as discrete linear dynamics

A Markov chain is a discrete-time linear system on distributions: $\pi_{t+1} = \pi_t P$, so $\pi_t = \pi_0 P^t$, the discrete analogue of $x(t) = e^{At}x(0)$.

- Stationary distribution $\pi^* = \pi^* P$ is the left eigenvector with $\lambda = 1$; it always exists.
- Convergence rate is the second eigenvalue: every other mode decays like $|\lambda_i|^t$, so $\pi_t \to \pi^*$ geometrically at rate $|\lambda_2|$.
- Under a positivity condition $\pi \mapsto \pi P$ is a contraction, so Banach gives a unique π^*.

References

- D. Acemoglu, *Introduction to Modern Economic Growth*, Appendix B.
- A. de la Fuente, *Mathematical Methods and Models for Economists*, Ch. 10.
- C. P. Simon and L. Blume, *Mathematics for Economists*, Ch. 24 and 25.