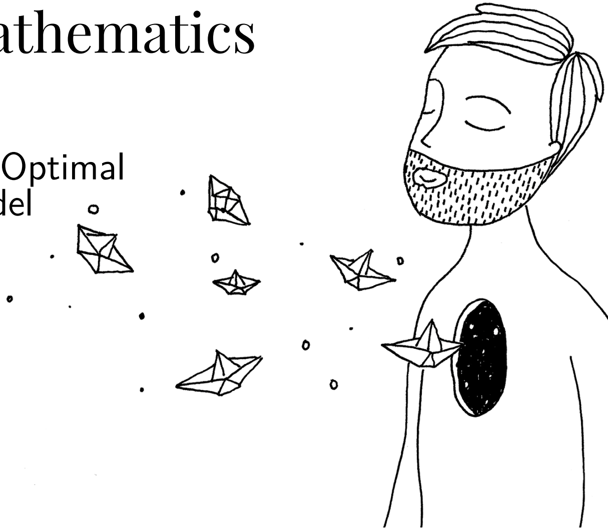
# 4509 - Bridging Mathematics

## Lecture 9: Continuous-Time Optimal Control and the Ramsey Model



PAULO FAGANDINI

NOVA SCHOOL OF BUSINESS & ECONOMICS

# Objectives

- Formulate continuous-time optimal control and **Pontryagin's Maximum Principle**.
- Master the **current-value Hamiltonian**, reading the costate $q(t)$ as a shadow price of the state.
- Derive the **Ramsey-Cass-Koopmans model**: the Keynes-Ramsey rule, the $(k, c)$ phase diagram, and saddle-path stability.

The spine: Pontryagin's principle gives a Hamiltonian whose costate $q(t)$ is a **shadow asset price** for the state. Apply it to growth and you get Ramsey: the Keynes-Ramsey rule (Euler in continuous time), a $(k, c)$ phase diagram (Lecture 7's recipe), and a **saddle path**, the unique non-explosive plan. Along it, consumption is the annuity of lifetime wealth.

# The control problem

Choose a path for the control $u(t) \in U$ to maximize discounted returns; the state $x(t)$ then follows from its law of motion and the initial condition:

$$\max_{\{u(t)\}} \int_0^\infty e^{-\rho t} F(x, u)\, dt \quad \text{s.t.} \quad \dot{x} = m(x, u),\ x(0) = x_0,\ \rho > 0,$$

with $F$ the instantaneous (current) utility and $m$ the transition. The current choice of $u$ acts on two margins: the **immediate** payoff through $F$, and the **future** through its effect on $\dot{x}$.

Strategy: a static maximization at each instant, tied together across time by a system of ordinary differential equations.

# The current-value Hamiltonian

Attach to each state a **costate** $q(t)$, the shadow price of $x$, and form the modified objective

$$\mathcal{H}^c(x,u,q) = \underbrace{F(x,u)}_{\text{current gains}} + q\,\underbrace{m(x,u)}_{\dot{x}}.$$

Read it as current gains plus the value, at price $q$, of the investment in the future carried by the change in the state. In present-value units the costate is $\lambda(t) = e^{-\rho t} q(t)$; working with $q$ keeps the discount factor out of the algebra.

# Necessary conditions

An interior optimum satisfies, at each $t$:

1. **Control (FOC):** $\dfrac{\partial \mathcal{H}^c}{\partial u} = F_u + q\, m_u = 0$, so $q$ reflects the marginal value of the state.
2. **Costate:** $\dot{q} = \rho q - \dfrac{\partial \mathcal{H}^c}{\partial x} = \rho q - \big[F_x + q\, m_x\big]$.
3. **State:** $\dot{x} = \dfrac{\partial \mathcal{H}^c}{\partial q} = m(x, u^*)$.

Here $\partial \mathcal{H}^c / \partial x = F_x + q\, m_x$ is the state's current marginal return $F_x$ plus its contribution to the stock $m_x$, valued at the shadow price $q$.

# The costate equation is an asset-pricing condition

Rearrange the costate equation as $\dfrac{\mathcal{H}_x^c + \dot{q}}{q} = \rho$, i.e.

$$\underbrace{\frac{\mathcal{H}_x^c}{q}}_{\text{dividend yield}} + \underbrace{\frac{\dot{q}}{q}}_{\text{capital gain}} = \underbrace{\rho}_{\text{required return}}.$$

Holding a unit of the state is like holding an **asset**: its total return, instantaneous dividend $\mathcal{H}_x^c$ plus capital gain $\dot{q}$ on the shadow price, must equal the required return $\rho$. Pontryagin's costate equation is a no-arbitrage condition. This single reframing is the point of the whole machinery for economists.

# Pontryagin's Maximum Principle

## Necessary conditions (infinite horizon)

Let $u^*(t)$ solve the problem. Then there **exist** continuous costate functions $q(t)$ such that, for every $t$:

1. $u^*(t) = \arg\max_u \mathcal{H}^c(x(t), u, q(t))$;
2. $\dot{x} = m(x, u^*)$ and $\dot{q} = \rho q - \mathcal{H}^c_x$;
3. transversality: $\lim\limits_{t\to\infty} e^{-\rho t} q(t) x(t) = 0$.

These are *existence* and *necessary* conditions, the continuous-time analogue of a first-order condition. The transversality condition rules out asymptotically over-accumulating a valuable state.

# From necessary to sufficient

Necessity alone leaves candidates that are not optima, exactly as in static optimization. Concavity closes the gap.

## Sufficiency

If the maximized Hamiltonian $\tilde{\mathcal{H}}(x, q) = \max_u \mathcal{H}^c(x, u, q)$ is concave in $x$ for given $q$, then any path meeting the necessary conditions and the transversality condition is **optimal**.

This is the "concavity makes the FOC enough" logic of the micro block, now along a whole path. When $x^*(t), q^*(t)$ converge to a steady state $(x^s, q^s) \geq 0$, that path is optimal, which is what makes the saddle path *the* solution, not merely a candidate.

# The Ramsey-Cass-Koopmans model: setup

The planner maximizes discounted lifetime utility subject to capital accumulation:

$$\max_{\{c\}} \int_0^\infty e^{-\rho t} \frac{c^{1-\sigma} - 1}{1-\sigma} \, dt \quad \text{s.t.} \quad \dot{k} = f(k) - c - (n+g+\delta)k, \;\; k(0) = k_0.$$

Here $k$ is the **state** and $c$ the **control**, both per effective worker, and $f$ is neoclassical ($f' > 0$, $f'' < 0$, Inada). The term $(n+g+\delta)k$ is **depreciation plus dilution**: capital per effective worker is thinned at rate $n+g$ by population and technology growth.

Form the current-value Hamiltonian, with costate $q$ the shadow price of capital:

$$\mathcal{H}^c = \underbrace{\frac{c^{1-\sigma} - 1}{1-\sigma}}_{\text{instantaneous utility}} + q\big[\underbrace{f(k) - c - (n+g+\delta)k}_{\dot{k}}\big].$$

# The Ramsey model: optimality conditions

Apply Pontryagin to $\mathcal{H}^c = \dfrac{c^{1-\sigma}-1}{1-\sigma} + q\big[f(k) - c - (n+g+\delta)k\big]$:

1. **Control (FOC):** $\dfrac{\partial \mathcal{H}^c}{\partial c} = c^{-\sigma} - q = 0 \;\Rightarrow\; \boxed{q = c^{-\sigma}}$, the shadow price equals marginal utility.
2. **Costate:** $\dfrac{\partial \mathcal{H}^c}{\partial k} = q\big[f'(k) - (n+g+\delta)\big]$, so

$$\dot{q} = \rho q - \frac{\partial \mathcal{H}^c}{\partial k} = q\big[\rho + n + g + \delta - f'(k)\big].$$

3. **State:** $\dot{k} = \dfrac{\partial \mathcal{H}^c}{\partial q} = f(k) - c - (n+g+\delta)k$.
4. **TVC:** $\lim_{t\to\infty} e^{-\rho t} q(t) k(t) = 0$, ruling out over-accumulation.

# The Ramsey model: the Keynes-Ramsey rule

Two equations carry the consumption dynamics: the FOC $q = c^{-\sigma}$ and the costate. Eliminate $q$.

**Step 1.** Log-differentiate the FOC in time. From $\ln q = -\sigma \ln c$,

$$\frac{\dot{q}}{q} = -\sigma \, \frac{\dot{c}}{c}.$$

**Step 2.** Read $\dot{q}/q$ off the costate equation:

$$\frac{\dot{q}}{q} = \rho + n + g + \delta - f'(k).$$

**Step 3.** Equate and solve for $\dot{c}/c$:

$$-\sigma \, \frac{\dot{c}}{c} = \rho + n + g + \delta - f'(k) \implies \boxed{\frac{\dot{c}}{c} = \frac{f'(k) - (n + g + \delta) - \rho}{\sigma}.}$$

Consumption grows when the return net of dilution $f'(k) - (n + g + \delta)$ beats impatience $\rho$; $\sigma$ sets the brake.

# Building the phase diagram

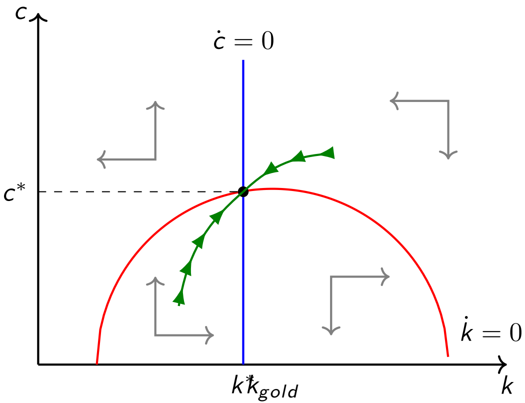
$$\begin{cases} \dot{k} = f(k) - c - (n+g+\delta)k, \\ \dot{c} = \dfrac{c}{\sigma}\big[f'(k) - (n+g+\delta) - \rho\big]. \end{cases}$$

Build it with Lecture 7's recipe:

- $\dot{c} = 0$: $f'(k^*) = n + g + \delta + \rho$, a **vertical line** at $k^*$; left of it $\dot{c} > 0$, right of it $\dot{c} < 0$.
- $\dot{k} = 0$: $c = f(k) - (n+g+\delta)k$, an **inverted U** peaking at the Golden Rule $k_{gold}$ where $f'(k_{gold}) = n + g + \delta$; above it $\dot{k} < 0$, below it $\dot{k} > 0$.

Sign the four regions; the intersection $(k^*, c^*)$ is the steady state. We verify next that it is a **saddle**, with a downward-sloping saddle path.

# The phase diagram




The two phase lines split the plane into four regions; the gray arrows compose the $\dot{k}$ and $\dot{c}$ motion. Only the two green branches converge to $(k^*, c^*)$: the **saddle path**.

# Verifying the saddle: the Jacobian

Linearize at $(k^*, c^*)$. Using $\dot{c} = 0 \Rightarrow f'(k^*) = n + g + \delta + \rho$ and $\dot{k} = 0 \Rightarrow c^* = f(k^*) - (n + g + \delta)k^*$,

$$J(k^*, c^*) = \begin{bmatrix} \frac{\partial \dot{k}}{\partial k} & \frac{\partial \dot{k}}{\partial c} \\ \frac{\partial \dot{c}}{\partial k} & \frac{\partial \dot{c}}{\partial c} \end{bmatrix} = \begin{bmatrix} \rho & -1 \\ \frac{c^*}{\sigma} f''(k^*) & 0 \end{bmatrix}.$$

The determinant is the product of the eigenvalues:

$$\det J = \lambda_1 \lambda_2 = \underbrace{\frac{c^*}{\sigma} f''(k^*)}_{<0 \text{ since } f''<0} < 0.$$

A negative determinant forces real eigenvalues of **opposite sign**: $(k^*, c^*)$ is a **saddle**. Exactly one trajectory, the stable manifold, converges; the jump in $c_0$ selects it. This is Lecture 7's classification, applied.

# The economics to say out loud

- **Consumption is the annuity of lifetime wealth.** With $r = \rho$ the household perfectly smooths and consumes the flow value of financial plus human wealth: you do not spend your wealth, you spend the return on it. The discrete cake-eating $c_t = (1-\beta)x_t$ was the same idea.
- **Keynes-Ramsey is tilt versus smooth.** If the return net of dilution $f'(k) - (n+g+\delta)$ beats impatience $\rho$, consumption rises; below, it falls; equal, it is flat. $\sigma$ is the smoothing brake.
- **The saddle path is the only sane choice.** From $k_0$, $c_0$ is a *jump*. Too high, you eat the capital and starve; too low, you over-accumulate and violate the TVC. Exactly one $c_0$ lands on the stable manifold.
- **Modified golden rule.** Impatient households stop *before* $k_{gold}$, at $f'(k^*) = n+g+\delta+\rho$. $\rho > 0$ is the whole reason $k^* < k_{gold}$.

# A worked comparative static

A permanent **fall in** $\rho$ (more patience) shifts the vertical $\dot{c} = 0$ line **right**, toward $k_{gold}$.

On impact $c$ **drops**, jumping down onto the new saddle path to fund faster accumulation; then the economy **climbs** the new path to higher long-run $(k^*, c^*)$.

This is exactly how one reasons about permanent shocks to preferences or technology in growth and macro-finance.

# References


- D. Acemoglu, *Introduction to Modern Economic Growth*, Appendix C and Ch. 8.
- A. de la Fuente, *Mathematical Methods and Models for Economists*, Ch. 11.
- D. Romer, *Advanced Macroeconomics*, Ch. 2 (Ramsey-Cass-Koopmans).